Diffraction limit of the sub-Planck structures

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The orthogonality of cat and displaced cat states, underlying Heisenberg limited measurement in quantum metrology, is studied in the limit of large number of states. The asymptotic expression for the corresponding state overlap function, controlled by the sub-Planck structures arising from phase space interference, is obtained exactly. The validity of large phase space support, in which context the asymptotic limit is achieved, is discussed in detail. For large number of coherent states, uniformly located on a circle, it identically matches with the diffraction pattern for a circular ring with uniform angular source strength. This is in accordance with the van Cittert-Zernike theorem, where the overlap function, similar to the mutual coherence function matches with a diffraction pattern.

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I. INTRODUCTION

Cat states and their generalizations are known to achieve Heisenberg limited sensitivity in estimation of parameters like coordinate/momentum displacements and phase space rotations [1]. A criterion to distinguish quantum states without classical counterparts, from those without the same, are studied in [2, 3]. For these non-classical states, subtle interference effects in the phase space [4] lead to sub-Planck structures in their Wigner functions, which in turn allow precision measurement of quantum parameters, bettering the standard quantum limit. Sub-Planck structures in different physical systems have been recently investigated [5–11]. It has been demonstrated [7–9] that the sensitivity of the state used in quantum metrology is directly related to the area of the sub-Planck structures: $\rho = \frac{\hbar^2}{A}$, with A being the action of the effective support of the Wigner function. The interference in phase space is a pure quantum phenomenon, arising due to the fact that these states are superposition of the coherent states (CSs), which themselves are classical. The increase in the number of interfering coherent states in the phase space is akin to emergence of diffraction in classical optics, when the number of interfering sources becomes large.

Here, we analyze this diffraction limit of the sub-Planck structures and find an exact asymptotic value of the displacement sensitivity. With the assumption of large phase-space support for the estimating state and smallness of the quantum parameters to be estimated, it is found that the asymptotic limit of the sensitivity reaches $|\delta| = \frac{C}{2|\alpha|}$, where C is the first root of J_0, the 0^{th} order Bessel function. We explicitly show that this assumption is adequate to adapt realistic values of the physical parameters; *i.e.*, the average photon number and the number of superposed CSs. The numerical analysis depicts how the asymptotic limit of exact overlap function (OF) reaches to the 0^{th} order Bessel function for higher order mesoscopic superpositions. This limiting behaviour in the phase space interference is found to be the exact analog of the van Cittert-Zernike theorem [12], relating the mutual coherence in classical optics to diffraction.

II. RESULTS AND DISCUSSIONS

Cat states and their generalizations play a significant role in quantum optics and quantum computation [13]. A number of experimental schemes exist to produce cat states in laboratory conditions [14]. These "pointer states"[15] often naturally manifest, when suitable quantum systems are coupled with decohering environment. It has been observed that the robustness of these states, made out of classical CSs, is a result of "quantum Darwinism" [16]. We consider a single oscillator, with the CS being an eigen state of a: $a|\alpha\rangle = \alpha|\alpha\rangle$, with annihilation and creation operator a and $a^\dagger$: $[a, a^\dagger] = 1$.

The generalized cat state is composed of CSs, equally phase displaced on a circle:

$$|cat_{n,\alpha}\rangle = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}|e^{\frac{\iota 2\pi j}{n}}\alpha\rangle = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}D(e^{\frac{\iota 2\pi j}{n}}\alpha)|0\rangle, \quad (1)$$

where, $|\alpha\rangle = D(\alpha)|0\rangle$, with the displacement operator, $D(\alpha) = e^{\alpha a^\dagger - \alpha^\star a}$ and $a|0\rangle = 0$.

The displacements in the coordinate and momenta can be realized through an appropriately displaced cat state [7]: $|cat^\delta_{n,\alpha}\rangle = D(\delta)|cat_{n,\alpha}\rangle$. For checking the sensitivity of the estimating state $|cat_{n,\alpha}\rangle$, one computes the overlap of the same with the displaced state and studies the

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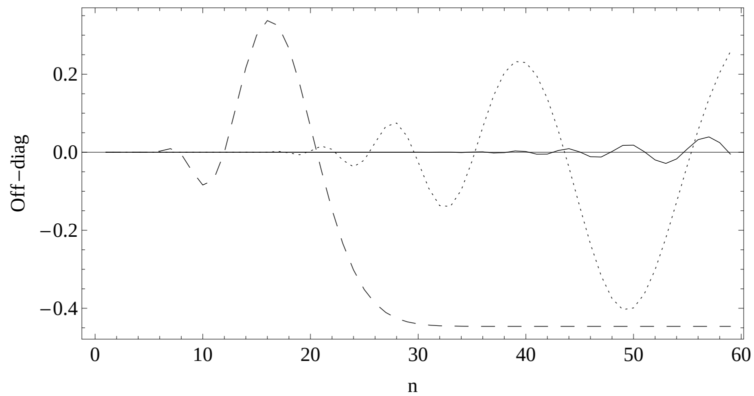

FIG. 1: Contribution of the off-diagonal terms ($j \neq k$) in Eq. (2) or (3)) for different phase space supports: $\alpha = 4$ (dashed line), $\alpha = 10$ (dotted line), $\alpha = 20$ (solid line), for an arbitrary fixed value of $\delta = 0.2$.

orthogonality conditions,

$$\begin{aligned}
\langle cat_{n,\alpha}|cat^{\delta}_{n,\alpha}\rangle &= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle 0|D(e^{\frac{i2\pi j}{n}}\alpha)^{\dagger}D(\delta)D(e^{\frac{i2\pi k}{n}}\alpha)|0\rangle \\
&= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(e^{iIm(\delta\alpha^{\star}(e^{-\frac{i2\pi j}{n}}+e^{-\frac{i2\pi k}{n}})+|\alpha|^2 e^{-\frac{i2\pi(k-j)}{n}})})(e^{-\frac{1}{2}|\delta+\alpha(e^{-\frac{i2\pi k}{n}}-e^{-\frac{i2\pi j}{n}})|^2}). \qquad (2) \\
&= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}e^{i(2rcos(\frac{\pi(j-k)}{n})\ sin(\theta-\frac{\pi(j+k)}{n})+|\alpha|^2\ sin(\frac{2\pi(j-k)}{n}))} \\
&\quad \times e^{-\frac{1}{2}(|\delta|^2+2|\alpha|^2(1-cos(\frac{2\pi(j-k)}{n}))+4r\ sin(\frac{\pi(j-k)}{n})sin(\theta-\frac{\pi(j+k)}{n}))} \qquad (3)
\end{aligned}$$

where, $r = |\alpha||\delta|$ and $\theta = (\theta_\delta - \theta_\alpha)$ with $\alpha = |\alpha|e^{i\theta_\alpha}$ and $\delta = |\delta|e^{i\theta_\delta}$.

The entire contribution of the OF mainly originates from the adjacent components of the original and displaced cat states, *i.e.*, $j \sim k$. Therefore, $|j-k| << n$, $cos(\pi(j-k)/n) \to 1$ and $sin(\pi(j-k)/n) \to 0$. Then Eq. (3) takes the simpler form

$$\langle cat_{n,\alpha}|cat^{\delta}_{n,\alpha}\rangle = \frac{e^{-\frac{1}{2}|\delta|^2}}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}cos\left[2rsin(\theta-\frac{\pi(j+k)}{n})\right]. \qquad (4)$$

Apparently, the off-diagonal terms in the above expression are only significant for higher order superpositions ($n >> 1$). Now, with the assumption of large phase-space support for the estimating state and smallness of quantum parameters to be estimated, one can neglect the off-diagonal terms and obtain

$$\begin{aligned}
\langle cat_{n,\alpha}|cat^{\delta}_{n,\alpha}\rangle &\approx \frac{e^{-\frac{1}{2}|\delta|^2}}{n}\sum_{j=1}^{n}cos\left[2rsin(\theta-\frac{2\pi j}{n})\right] \\
&\approx \frac{1}{n}\sum_{j=1}^{n}cos\left[2rsin(\theta-\frac{2\pi j}{n})\right]. \qquad (5)
\end{aligned}$$

The phase space of the generalized cat state of Eq. 1 is composed of 'n' CSs, equally placed in a circle of radius $|\alpha|$, where large phase space support means the large magnitude of $|\alpha|$.

In this context, we must emphasize more on the domain of validity of the above assumption. Fig. 1 depicts the contribution of the off-diagonal terms of Eq. (2) or Eq. (3). It becomes significant only after very large value of 'n' or when n/α ratio is approximately two for a fixed phase space support. Increasing n or producing a higher order superposition is quite difficult in experiments, as it requires a large nonlinearity of the medium. On the contrary, the absolute value of α is directly related to the average photon number of the coherent state, which can

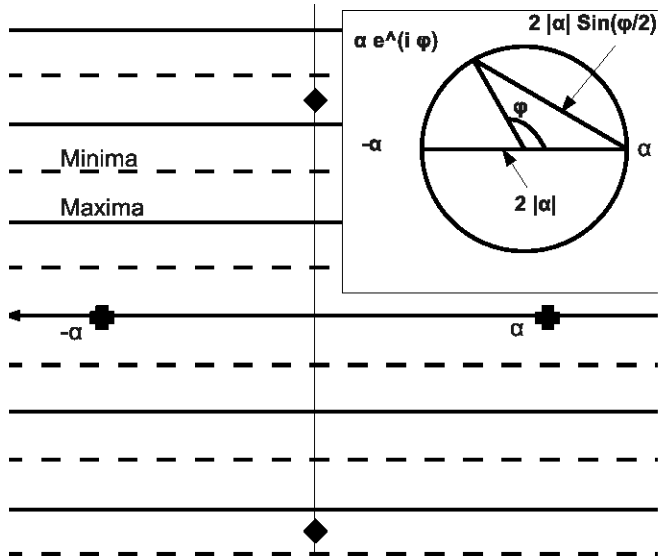

FIG. 2: **Analogy between two-source interference:** The solid lines show the maximum and dashed lines show the minimum intensity values. The crosses are the positions of coherent state and diamonds show the equivalent positions of sources of light which will produce the same pattern at a distance. The inset shows the equivalent position of sources for the state $|cat2_{\alpha,\phi}\rangle = \frac{|\alpha\rangle + |\alpha e^{\iota\phi}\rangle}{2}$

be manipulated by controlling the laser beam. Hence, the allowed maximum order of mesoscopic superposition ('n') for a given α, conforming our assumption, is sufficiently large in reality.

It needs to be mentioned that the state overlap depends only on $\delta\alpha^\star$, which leads to the conclusion that the sensitivity of estimating δ is inversely proportional to $|\alpha|$. For convenience, we assume n is even:

$$\langle cat_{n,\alpha}|cat^{\delta}_{n,\alpha}\rangle = \frac{2}{n}\sum_{j=1}^{\frac{n}{2}} \cos\left[2r\sin(\theta - \frac{2\pi j}{n})\right]. \tag{6}$$

It is easily checked that the OF, being of interferometric origin, is only sensitive to the difference in phase : $\langle cat^{\delta_2}_{n,\alpha}|cat^{\delta_1}_{n,\alpha}\rangle = e^{\iota\phi}\langle cat_{n,\alpha}|cat^{\delta_1-\delta_2}_{n\alpha}\rangle$. The OF for $n=2$,

$$|\langle cat_{2,\alpha}|cat^{\delta}_{2,\alpha}\rangle|^2 = \cos^2(2|\alpha|\delta_\perp) \tag{7}$$

matches with the known result [7], with $\delta_\perp = |\delta|\sin(\theta_\delta - \theta_\alpha)$ and $\delta_\parallel = |\delta|\cos(\theta_\delta - \theta_\alpha)$. As is depicted in Fig. 2, it is interesting to observe that the above expression is analogous to the double slit interference pattern, where the normalized intensity can be written as $\frac{I}{I_{max}} = \cos^2(\frac{yb\pi}{s\lambda})$ [17]. The path difference between the two waves reaching at the observation point is yb/s, where b defines the distance between the two slits, s is the separation between the aperture and the screen, and y corresponds to the vertical coordinate of the detector. The above analogy can be mathematically established by taking λ in the unit of s and redefining the commutation relation, $[a, a^\dagger] = \pi\lambda^{-1}$:

$$|\langle cat_{2,\alpha}|cat^{\delta}_{2,\alpha}\rangle|^2 = \cos^2\left[2\frac{|\alpha|\delta_\perp\pi}{\lambda}\right], \tag{8}$$

where $2|\alpha|$ is the separation of the two coherent state sources. Use of the phase shifted cat state, $|cat2_{\alpha,\phi}\rangle = \frac{|\alpha\rangle + |\alpha e^{\iota\phi}\rangle}{2}$, would yield an interference pattern at an angle $\frac{\phi}{2}$ and *fringe width*, $2|\alpha|\sin\frac{\phi}{2}$:

$$|\langle cat2_{\alpha,\phi}|cat2^{\delta}_{\alpha,\phi}\rangle|^2 = \cos^2(2|\alpha|\sin\frac{\phi}{2}(\delta_\perp\sin\frac{\phi}{2} + \delta_\parallel\cos\frac{\phi}{2})) \tag{9}$$

Introducing a phase between the constituent CSs of a cat state with $n=2$ gives the state $|cat2^{\phi}_{\alpha}\rangle = \frac{|\alpha\rangle + e^{\iota\phi}|-\alpha\rangle}{2}$. The OF for this state is

$$|\langle cat2^{\phi}_{\alpha}|cat2^{\phi,\delta}_{\alpha}\rangle|^2 = \cos^2(2|\alpha|\delta_\perp) - \phi), \tag{10}$$

akin to the phenomenon of "*fringe shift*" observed in classical optics.

We now derive the asymptotic limit to the state overlap and sensitivity in parameter estimation,

$$\begin{aligned}
\varinjlim_{n\to\infty} \langle cat_{n,\alpha}|cat^{\delta}_{n,\alpha}\rangle &= \varinjlim_{n\to\infty} \frac{2}{n}\sum_{j=1}^{\frac{n}{2}} \cos(2r\sin(\theta - \frac{2\pi j}{n})) \\
&= \varinjlim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \cos(2r\sin(\theta - \frac{2\pi j}{n})) \\
&= \int_0^1 \cos(2r\sin(\theta - 2\pi x))dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(2r\sin(z))dz \\
&= J_0(2|\alpha||\delta|).
\end{aligned} \tag{11}$$

This proves our assertion that states can be discriminated for $|\delta| = \frac{C}{2|\alpha|}$ due to orthogonality, where C is a root of the Bessel function (of first kind) of order zero, *i.e.*, J_0.

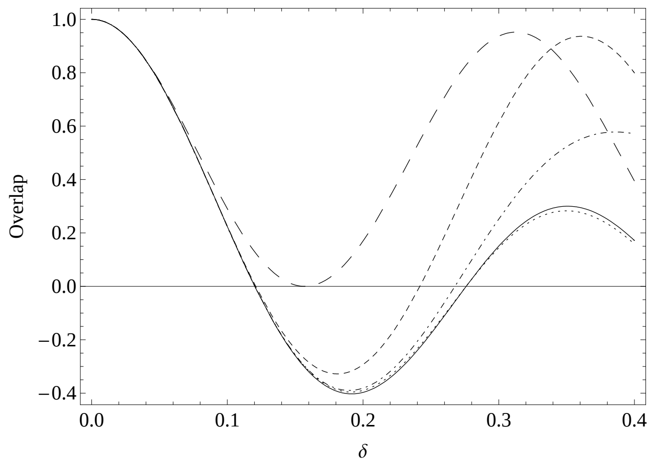

FIG. 3: Overlap function (Eq. (2) or (3)) for superposition of four coherent states (dashed line), superposition of six coherent states (small-dashed line), superposition of eight coherent states (dot-dashed line), and superposition of sixteen coherent states (dotted line). For larger value of 'n', the OF gradually coincides with the zeroth order Bessel function (solid line). Here, $\alpha = 10$ for all the cases.

Although we have already discussed about the reliability of our assumption before (see Fig. 1), we want to further check, whether the above result is still valid for the general OF in Eq. (2) or (3). We provide Fig. 3 for numerical delineation, which clearly shows the agreement of the zeroth order Bessel function for higher order mesoscopic superposition. Hence, the experimental parameter domain is well in the range of our previous assumption, where Eq. (11) is valid.

The asymptotic overlap function [Eq. (11)] is the result of coherent superposition of $n-$CSs situated in a ring of radius α. For $n \to \infty$, the ring behaves as a ringed shaped light source with constant angular source strength. Hence. the superposition is analogous to the diffraction pattern generated when light passes through the thin ring shaped opening. The fact that the overlap between the cat states and their shifted forms, is of the same form as the diffraction pattern centered at one of the states, bears strong resemblance to the van Cittert-Zernike theorem [12], where the normalized mutual coherence function $\gamma_{12}(0)$ between two points is identical to diffraction pattern centered at one of the points. For a ring shaped opening with constant angular source strength, one can write explicitly

$$\begin{aligned}\gamma_{12}(0) &= \frac{\langle E_1(t)E_2(t)^*\rangle_T}{\sqrt{\langle E_1(t)E_1(t)^*\rangle_T \langle E_2(t)E_2(t)^*\rangle_T}} \\ &= J_0\left(\frac{2\pi r_0|\vec{r_1}-\vec{r_2}|}{\lambda R}\right). \end{aligned} \qquad (12)$$

$\gamma_{12}(0)$ actually signifies the complex degree of spatial coherence of the two points at the same instant in time, when fields arriving at the observation screen being $E_1(t)$ and $E_2(t)$ respectively. r_0 is the radius of ring, R is the distance of the screen from the opening and $|\vec{r_1}-\vec{r_2}|$ is the path difference between the points. The suffix T in the expectation value signifies the time average according to the ergodic hypothesis. The above equation should be compared with the OF for large n (Eq. 11), for unit distance from the screen to the opening ($R = 1$) and for $[a, a^\dagger] = \pi\lambda^{-1}$:

$$\langle cat_{n,\alpha}|cat^{\delta}_{n,\alpha}\rangle = J_0\left(\frac{2\pi|\alpha||\delta|}{\lambda}\right). \qquad (13)$$

Thus the OF is similar to the mutual coherence function, nicely matches with the diffraction pattern in accordance with the van Cittert-Zernike theorem.

Considering the sinusoidal nature of the OF, it needs to be checked where the states are not distinguishable. The fact that:

$$\langle cat_{n,\alpha}|cat^{\delta}_{n,\alpha}\rangle = \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle 0|D(e^{\frac{\iota 2\pi j}{n}}\alpha)^\dagger D(\delta)D(e^{\frac{\iota 2\pi k}{n}}\alpha)|0\rangle \qquad (14)$$

indicates that the maximum contribution of each term is $\frac{1}{n}$ and minimum $\frac{-1}{n}$, which arise, if and only if, either $D(\alpha)$ and $D(\delta)$ commute($\frac{1}{n}$) or anti- commute($\frac{-1}{n}$). Thus, the state is indistinguishable from its shifted form, if and only if, displacement operators corresponding to all the constituent states of the cat state commute with $D(\delta)$ or all anti- commute with it.

III. CONCLUSIONS

In conclusion, the sensitivity of cat-like states to quantum parameter estimation is studied for large number of constituent CSs. The assumption of large phase space support is justified for accessible parameter ranges in realistic situation. In this limit, the state OF, determining the orthogonality of cat and displaced cat states, approaches the Bessel function. According to the van Cittert-Zernike theorem, the coherence problem is mathematically identical with the diffraction problem by complex degree of coherence. The fact that the OF is having the same form as the diffraction pattern results the same expression of normalized mutual coherence function for large n. This is similar to the mutual coherence function of a circular ring, which yields Bessel function of order zero, matching with the theorem of van Cittert-Zernike.

IV. ACKNOWLEDGMENT

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